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# PERTURBATIONS FROM A SOURCE IN A THREE-LAYER ATMOSPHERE†

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The problem of internal waves, excited by a point source in a three-layer, initially unperturbed, atmosphere is investigated in a linear formulation. It is assumed that the vertical displacements and the velocities of the particles change continuously at the boundaries of the layers and that the Brunt–Väisälä frequency is constant in each layer but suffers discontinuities at the boundaries of the layers. The solution, found using integral transforms, is expressed in terms of double integrals of multiple-valued analytic functions. The integral representation for the perturbations in the middle layer does not enable asymptotic methods to be used directly to obtain an approximate description of the behaviour of the solution at long times. It is transformed into finite sums of single-valued integrals which, in a certain sense, represent the various modes of oscillation which arise. Modes making the major contribution to the perturbation are investigated by the stationary phase method. Particular surfaces are found in the neighbourhood of which the amplitudes of the oscillations decay weakly with time. A problem on perturbations from a source in a two-layer atmosphere was investigated earlier in [1]. A study of the case of a three-layer atmosphere is of interest since the middle layer acts as a waveguide. © 2002 Elsevier Science Ltd. All rights reserved.

## 1. FORMULATION OF THE PROBLEM AND THE INTEGRAL REPRESENTATION OF THE SOLUTION

An ideal atmosphere which fills a three-dimensional space is considered. This atmosphere is subdivided into three layers with constant, but different, Brunt–Väisälä frequencies. The Brunt–Väisälä frequencies are $N_1$, $N_2$ and $N_3$ in the lower, middle and upper layers respectively. The width of the middle layer was chosen as the unit of length and $1/N_2$ as the unit of time. It is assumed that $N_1/N_2 = a < 1$ and $N_3/N_2 = a' < 1$. The origin of the Cartesian system of coordinates, $xyz$, is chosen to be on the lower boundary of the middle layer and the $z$ axis is directed opposite to the gravity force. The source is located in the lower layer at the point $(0, 0, -c)$. The power of the source $Q(t)$ is a continuously differentiable function and $Q(0) = Q'(0) = 0$. The vertical deviation $W$ of the fluid particles from the equilibrium position is expressed in terms of the derivative $\partial w/\partial z$, where the function $w(x, y, z, t)$ is the solution of the following problem

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}(\Delta w) + \Delta_2 w = 0, \quad 0 < z < 1 \\ &\frac{\partial^2}{\partial t^2}(\Delta w) + a^2 \Delta_2 w = \frac{Q'(t)}{4\pi}\delta(x)\delta(y)\delta(z+c), \quad -\infty < z < 0 \qquad (1.1) \\ &\frac{\partial^2}{\partial t^2}(\Delta w) + a'^2 \Delta_2 w = 0, \quad 1 < z < +\infty \end{aligned}$$

The function $w$ and its first-order partial derivatives are bounded and change continuously on crossing the boundaries of the layers. The initial conditions are null conditions.

Applying a Laplace transformation with respect to the variable $t$ with null initial conditions and a Hankel transformation with respect to $r$, we will seek a solution of problem (1.1) in the form

$$w = \frac{1}{16\pi^3 i}\int_0^{+\infty}\int_{C-i\infty}^{C+i\infty} \varphi(u, p, z) J_0(ru) \mathbf{L}Q(p) e^{pt} du \frac{dp}{p} \qquad (1.2)$$

where $\mathbf{L}Q$ is the Laplace transform of the function $Q$.

For determining the function $\varphi$, we obtain the equations

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$$\frac{d^2\varphi}{dz^2} - \omega^2\varphi = 0, \quad 0 < z < 1, \quad \omega = \frac{u}{p}\sqrt{1+p^2}$$

$$\frac{d^2\varphi}{dz^2} - \beta^2\varphi = 0, \quad 1 < z < +\infty, \quad \beta = \frac{u}{p}\sqrt{a'^2+p^2} \tag{1.3}$$

$$\frac{d^2\varphi}{dz^2} - \gamma^2\varphi = u\delta(z+c), \quad z < 0, \quad \gamma = \frac{u}{p}\sqrt{a^2+p^2}$$

Any solution of the second of Eqs (1.3), which is bounded when $z > 1$, satisfies the conditions $\varphi'(z)-\beta\varphi(z) = 0$. Any solution of the third equation of (1.3), which is bounded when $z < 0$, has the form $\varphi = -(u/2\gamma)e^{-\gamma|z+c|} + Ce^{z\gamma}$ and, when $z > -c$, satisfies the condition $\varphi'(z) - \gamma\varphi = ue^{-\gamma(z+c)}$.

Since the function $\varphi'(z)$ is continuous at the points $z = 0$ and $z = 1$, in order to determine the function $\varphi(z)$ in the interval [0, 1], it is necessary to solve the boundary-value problem

$$\frac{d^2\varphi(z)}{dz^2} - \omega^2\varphi(z) = 0$$

$$\frac{d\varphi(0)}{dz} - \gamma\varphi(0) = ue^{-\gamma c}, \quad \frac{d\varphi(1)}{dz} + \beta\varphi(1) = 0$$

The solution of this boundary-value problem has the form

$$\varphi(u,p,z) = -ue^{-\gamma c}\frac{(\omega+\beta)e^{\omega(1-z)} + (\omega-\beta)e^{-\omega(1-z)}}{(\omega+\beta)(\omega+\gamma)e^{\omega} - (\omega-\beta)(\omega-\gamma)e^{-\omega}}$$

The solution of problem (1.1) is obtained by substituting the function $\varphi(u, p, z)$ into formula (1.2). The case when $a' = a$, and consequently $\beta = \gamma$, is investigated later, namely

$$\varphi(u,p,z) = -ue^{-\beta c}\frac{(\omega+\beta)e^{\omega(1-z)} + (\omega-\beta)e^{-\omega(1-z)}}{(\omega+\beta)^2e^{\omega} - (\omega-\beta)^2e^{-\omega}} \tag{1.4}$$

## 2. TRANSFORMATION OF THE FORMULA GIVING THE SOLUTION

Since the function $\varphi(u, p, z)$, which is defined by equality (1.4), does not change its form when $\omega$ is replaced by $-\omega$, the points $p = \pm i$ are not branching points of this function. In the case of the function $\varphi(u,p,z)$, the two branching points are $p = \pm ai$. We join these points by a cut along the imaginary axis and we will show that the poles of the integrand lie in the sections $\pm[ai, i]$. These poles must be zeros of the equation $e^{2\omega} = (\omega-\beta)^2/(\omega + \beta)^2$. If $p$ is a zero of this equation, then it also follows from formula (1.3) that $-p$ and $\bar{p}$ are also zeros of this equation. It is therefore sufficient to consider the case when the point $p$ lies in the first quadrant. In this case, $0 < \arg\sqrt{1 + p^2} < \arg p < \pi/2$ and, consequently, the point $\omega$ lies in the fourth quadrant. Similarly, the point $\beta$ lies in the fourth quadrant. But then $|\omega - \beta|/|\omega + \beta| < 1$, $|e^{\omega}| > 1$, and the equality $|e^{\omega}| = |\omega - \beta|/|\omega + \beta|$ cannot be satisfied. So, the poles of the integrand can only lie on the imaginary axis. Moreover, they are symmetrically disposed about the origin of coordinates. These poles are the zeros of the equations

$$\operatorname{cth}\omega = -\omega/\beta, \quad \operatorname{th}\omega = -\omega/\beta \tag{2.1}$$

When $p = iq$, $q > 1$, the numbers $\omega$ and $\beta$ are positive and Eqs (2.1) only have a null solution. It is seen from formulae (1.3) and (1.4) that zero is not a pole of the integrand. If $-a < q < a$, then $\omega$ and $\beta$ are pure imaginary and Eqs (2.1) do not have solutions. If, however, $a < q < 1$, then $\omega$ will be an imaginary number and $\beta$ will be a real number, and Eqs (2.1) have solutions.

Using Cauchy's theorem, we transform formulae (1.2) to the form

$$w = w_1 + w_2 + w_3 \tag{2.2}$$

$$w_k = \frac{1}{16\pi^3 i}\int_0^{+\infty}\int_{C_k} J_0(ru)\mathbf{L}Q(p)e^{pt}\varphi(u,p,z)\frac{dp}{p}\,du$$

where the contour $C_1$ is the double section $[-ai, ai]$, the circle $C_2$ contains inside it all the zeros of Eqs (2.1) lying in the section $[ai, i]$ and the circle $C_3$ is symmetrical with $C_2$ about the real axis and contains the zeros of Eqs (2.1) lying in the section $[-i, -ai]$. The number $p$ takes complex-conjugate values at points of the circles which are symmetrical about the real axis. It follows from formulae (1.3) and (1.4) that $\varphi(u,\bar{p},z) = \overline{\varphi(u,p,z)}$. Noting that, when the point $p$ moves around the circle $C_2$ in a positive direction, the conjugate point $\bar{p}$ moves around the circle $C_3$ in the opposite direction, we obtain from formula (2.2)

$$w_2 + w_3 = \frac{1}{8\pi^3}\operatorname{Im}\int_0^{+\infty}\int_{C_2} J_0(ru)\mathbf{L}Q(p)e^{pt}\varphi(u,p,z)\frac{dp}{p}du \tag{2.3}$$

Introducing the new variables

$$p = iq, \quad x = \frac{u}{q}\sqrt{1-q^2}, \quad b^2 = 1-a^2, \quad u = \frac{av}{b} \tag{2.4}$$

we can represent expression (2.3) in the form

$$w_2 + w_3 = \sum_{k=1}^{2} \operatorname{Im}\int_0^{+\infty} \frac{1}{2\pi i}\int_{C_k} \frac{\Phi_1(r,x,v)\Phi_2(1-z,x,v)dxdv}{\psi_1(x,v)\psi_2(x,v)} \tag{2.5}$$

$$\Phi_1(r,x,v) = J_0\left(\frac{rav}{b}\right)\mathbf{L}Q\left(\frac{iav}{E(x,v)}\right)\exp\left(\frac{itav}{E(x,v)}\right)$$

$$\Phi_2(z,x,v) = \frac{ia^2vx}{8\pi^2}e^{-ac\sqrt{v^2-x^2}}\,\frac{x\cos xz + a\sqrt{v^2-x^2}\sin xz}{E^2(x,v)}$$

$$\psi_1(x,v) = x\cos\frac{x}{2} + a\sqrt{v^2-x^2}\sin\frac{x}{2}$$

$$\psi_2(x,v) = x\sin\frac{x}{2} - a\sqrt{v^2-x^2}\cos\frac{x}{2}, \quad E(x,v) = \sqrt{(av)^2+(bx)^2}$$

Graphs of the functions $-a\operatorname{tg}(x/2)$ and $x/\sqrt{v^2-x^2}\,a\operatorname{ctg}(x/2)$ show that, for each fixed $v$, the functions $\psi_1$ and $\psi_2$ have a finite number of zeros $\pm x_k(v)$ ($k = 0, \ldots, 2N(v) + 1$) and, moreover, the zeros of these functions alternate, being increasing functions of the parameter $v$. Suppose $\psi_1(x_{2k+1}(v), v) = 0$ and $\psi_2(x_{2k}(v, v) = 0$. The functions $x_{2k}(v)$ are defined in the interval $2\pi k < v < +\infty$ and $2\pi k < x_{2k}(v) < (2k+1)\pi$. The functions $x_{2k+1}(v)$ are defined in the interval $(2k+1)\pi < v < +\infty$ and $(2k+1)\pi < x_{2k+1}(v) < (2k+2)\pi$.

On replacing the inner integrals in formula (2.5) by the sum of the residues, we obtain

$$w_2 + w_3 = \operatorname{Re}\frac{1}{8\pi^2}\left(\sum_{k=0}^{N(v)}\int_0^{+\infty} F_1(x_{2k+1}(v),r,z,t)x'_{2k+1}(v)dv \; + \right.$$

$$\left. + \operatorname{Re}\sum_{k=0}^{N(v)}\int_0^{+\infty} F_2(x_{2k}(v),r,z,t)x'_{2k}(v)dv\right) \tag{2.6}$$

where

$$F_1(x,r,z,t) = J_0\left(\frac{rxA(x)}{b\,|\sin(x/2)|}\right)\mathbf{L}Q(iA(x))\exp\left(itA(x) - cx\left|\operatorname{ctg}\frac{x}{2}\right|\right)\cos\frac{x}{2}\sin x\left(z-\frac{1}{2}\right)$$

$$F_2(x,r,z,t) = J_0\left(\frac{rxB(x)}{b\,|\cos(x/2)|}\right)\mathbf{L}Q(iB(x))\exp\left(itB(x) - cx\left|\operatorname{tg}\frac{x}{2}\right|\right)\sin\frac{x}{2}\cos x\left(z-\frac{1}{2}\right)$$

$$A(x) = \sqrt{1-(1-a^2)\sin^2\frac{x}{2}}, \quad B(x) = \sqrt{1-(1-a^2)\cos^2\frac{x}{2}}$$

and the variable $v$ is related to the variable $x$ by formulae (2.4).

Whenever the functions $x_n(v)$ are undefined, we continue them with a zero. If the monotonicity of the functions $x_n(v)$ is used and the variable of integration $x = x_n(v)$ is replaced in each of the integrals of formula (2.6), then formula (2.6) can be written in the form

$$w_2 + w_3 = \mathrm{Re}\frac{1}{8\pi^2}\int_0^\pi F_2(x+2k\pi, r, z, t)dx + \mathrm{Re}\frac{1}{8\pi^2}\int_0^\pi F_1(x+(2k+1)\pi, r, z, t)dx \qquad (2.7)$$

Using expressions (2.6), formula (2.7) can be written in the form

$$w_2 + w_3 = -\mathrm{Im}\frac{1}{16\pi^2}\int_0^\pi B(x)\sin\frac{x}{2}\mathbf{L}Q(iB(x))e^{itB(x)-cx\,\mathrm{tg}(x/2)}\Psi(x, r, z)dx \qquad (2.8)$$

$$\Psi(x, r, z) = \sum_{k=0}^{\infty} e^{-c\pi k\,\mathrm{tg}(x/2)}\cos\left(x\left(z-\frac{1}{2}\right)+k\pi z\right)J_0\left(\frac{r(x+k\pi)B(x)}{b\,|\cos(x/2)|}\right) \qquad (2.9)$$

When $x = 0$, we obtain, using well-known identities [2]

$$\Psi(0, r, z) = \sum_{k=0}^{\infty} J_0\left(\frac{k\pi a}{b}r\right)\cos(k\pi z) = \frac{1}{2}+\frac{1}{\pi}\sum_{l=m+1}^{n} G(r, 2l-z), \quad \text{if} \quad bz > r$$

$$\Psi(0, r, z) = \frac{1}{2}+\frac{1}{\pi}\sum_{l=1}^{m}(G(r, 2l+z)+G(r, 2l-z)), \quad \text{if} \quad \frac{bz}{r} < 1$$

$$G(r \pm lz) = \frac{b}{\sqrt{a^2r^2 - b^2(2l \pm z^2)}}, \quad m = \left[\frac{|ar-bz|}{2b}\right], \quad n = \left[\frac{bz+ar}{2b}\right]$$

where the integer part of the number $x$ is denoted by $[x]$.

We will investigate the behaviour of integral (2.8) for large values of $t$. The phase $B(x)$ has a stationary point at the ends of the interval of integration. The contribution from the point $x = \pi$ decreases more rapidly than any negative power of $t$ and the contribution from the point $x = 0$ is equal to

$$w_1 + w_2 = \mathrm{Re}\frac{a\mathbf{L}Q(ia)\Psi(0, r, z)e^{ita}}{128\pi^2b^2t} \qquad (2.10)$$

For fixed values of $r$ and $z$, the function $w_1 + w_2$ decreases as $e^{ita}/t$. It follows from formulae (2.9) and (2.10) that, in the neighbourhood of particular surfaces (cones) $a^2r^2 - b^2(2l\pm z)^2 = 0$, the function $\Psi(0, r, z)$ becomes unbounded and, consequently, close to these surfaces the oscillations will slowly decay.

We will investigate the behaviour of the term $w_1$ in formula (2.2). Making the change of variables

$$p = qi \pm 0, \quad \omega(qi \pm 0, u) = -i\omega_1, \quad \omega_1 = u\frac{\sqrt{1-q^2}}{q}$$

$$\beta(qi \pm 0, u) = \mp i\beta_1, \quad \beta_1 = u\frac{\sqrt{a^2-q^2}}{q}$$

we transform the expression for $w_1$ to the form

$$w_1 = -\frac{1}{4\pi^3}\int_0^{+\infty}\int_0^a J_0(ru)\mathrm{Re}\left(\mathbf{L}Q(iq)e^{iqt}\right)\mathrm{Re}\,F(\omega_1, \beta_1, z)du\,dq \qquad (2.11)$$

$$F(\omega_1, \beta_1, z) = u\left(e^{i\beta_1 c - i\omega_1 z} + \frac{\omega_1-\beta_1}{\omega_1+\beta_1}e^{i\beta_1 c - \omega_1(2-z)}\right)\left(q(\omega_1+\beta_1)\left(1-\left(\frac{\omega_1-\beta_1}{\omega_1+\beta_1}e^{-i\omega_1}\right)^2\right)\right)^{-1}$$

Putting

$$\frac{\omega_1-\beta_1}{\omega_1+\beta_1} = \frac{1}{b^2}\left(\sqrt{1-q^2}-\sqrt{a^2-q^2}\right)^2 = s^2(q)$$

$$\frac{u}{q(\omega_1+\beta_1)}=\frac{1}{b}s(q)$$

we rewrite formula (2.11) in the form

$$w_1=-\frac{1}{4\pi^3 b}\sum_{k=0}^{\infty}\int_0^a s^{4k+1}\,\mathrm{Re}\left(\mathbf{L}Q(iq)e^{iqt}\right)\times$$

$$\times\ \mathrm{Re}\int_0^{+\infty} J_0(ru)\left(e^{i\beta_1 c-i\omega_1(z+2k)}+s^2 e^{i\beta_1 c-i\omega_1(2-z+2k)}\right)du\,dq$$

Evaluating the inner integrals, we obtain

$$w_1=-\frac{1}{4\pi^3 b}\sum_{k=0}^{\infty}\int_0^a s^{4k+1}\,\mathrm{Re}\left(\mathbf{L}Q(iq)e^{iqt}\right)\left(\sigma(r,z+2k)+s^2\sigma(r,2-z+2k)\right)dq \tag{2.12}$$

$$\sigma(q,r,z)=\int_0^{+\infty} J_0(ru)\exp(iuA)du=\frac{\theta(r^2-A^2)}{\sqrt{r^2-A^2}}$$

$$A(q,z)=\frac{1}{q}\left(z\sqrt{1-q^2}-c\sqrt{a^2-q^2}\right),\quad \theta(x)=\begin{cases}1 & \text{for } x\geqslant 1\\ 0 & \text{for } x<0\end{cases}$$

After replacing the variable of integration

$$q(x)=\sqrt{\frac{a^2-x^2}{1-x^2}},\quad s(q)=\sqrt{\frac{1-x}{1+x}}$$

formula (2.12) reduces to the form

$$w_1=-\frac{b}{4\pi^3}\sum_{k=0}^{\infty}\int_0^a \frac{x(1-x)^{2k-1}}{(1+x)^{2k+2}}\mathrm{Re}\left(\mathbf{L}Q(iq(x))e^{iq(x)t}\right)\times$$

$$\times\ \frac{\sigma(q(x),r,z+2k)+s^2\sigma(q(x),r,2-z+2k)}{\sqrt{a^2-x^2}}dx \tag{2.13}$$

where the function $\sigma(q(x), r, z)$ is non-zero only if

$$z<\frac{a}{b}L(r),\quad \max\{0,x_1\}\leqslant x\leqslant x_2 \tag{2.14}$$

$$x_{1,2}=\frac{b^2cz\mp r\sqrt{a^2L^2(r)-b^2z^2}}{L^2(r)},\quad L(r)=\sqrt{b^2c^2+r^2}$$

In this case, the equality

$$\frac{\sigma(q(x),r,z)}{\sqrt{a^2-x^2}}=\frac{1}{L(r)\sqrt{(x-x_1)(x_2-x)}}$$

holds.

Putting

$$\frac{bz}{aL(r)}=\sin\varphi,\quad \frac{cb}{L(r)}=\cos\psi,\quad \frac{r}{L(r)}=\sin\psi$$

we obtain $x_{1,2}=a\sin(\varphi\mp\psi)$.

When $z<ar/b$, both roots are real and $x_1\leqslant 0\leqslant x_2\leqslant a$. When $ar\leqslant z\leqslant aL(r)/b$, the roots are real and $0\leqslant x_1\leqslant x_2\leqslant a$ and, when $z=aL(r)/b$, the roots are identical: $x_1=x_2=abc/L(r)$. We further note that the root $x_2=a$ in the plane $z=ca$.

Suppose $N(r, z)$ is the largest number for which the inequality $z + 2k < ar/b$ holds and $M(r, z)$ is the largest number for which the inequality $ar/b < z + 2k < aL(r)/b$ holds. Equality (2.13) can be rewritten in the form

$$w_1 = -\mathrm{Re}\sum_{k=0}^{M(r,z)}\sum_{m=1}^{2}\int_0^{x_m(r,z+2k)} \frac{G_1(x,r,z+2k)e^{iq(x)t}dx}{\sqrt{(x-x_1(r,z+2k))(x_2(r,z+2k)-x)}} -$$

$$-\mathrm{Re}\sum_{k=0}^{M(r,2-z)}\sum_{m=1}^{2}\int_0^{x_m(r,2-z+2k)} \frac{G_2(x,r,2-z+2k)e^{iq(x)t}dx}{\sqrt{(x-x_1(r,2-z+2k))(x_2(r,2-z+2k)-x)}}$$

$$G_1(x,r,z) = \frac{b}{4\pi^3\sqrt{b^2c^2+r^2}}\frac{x(1-x)^{2k-1}}{(1+x)^{2k+2}}\mathbf{L}Q\left(i\sqrt{\frac{a^2-x^2}{1-x^2}}\right)$$

$$G_2(x,r,z) = \left(\frac{1-x}{1+x}\right)^2 G_1(x,r,z)$$

We will find the contributions to the asymptotic expansion of the integral from the points $x_1$ and $x_2$ [3]

$$w_1 = \mathrm{Re}\left(\sum_{k=0}^{M(r,z)}(-H_{12}(r,z+2k)) + \sum_{k=N(r,z)+1}^{M(r,z)} H_{11}(r,z+2k)\right) +$$

$$+\ \mathrm{Re}\left(\sum_{k=0}^{M(r,2-z)}(-H_{22}(r,2-z+2k)) + \sum_{k=N(r,2-z)+1}^{M(r,2-z)} H_{21}(r,2-z+2k)\right) \tag{2.15}$$

$$H_{ij}(r,z) = \frac{G_i(x_j(r,z),r,z)\exp(itq(x_j(r,z)+(-1)^j\pi/4))}{\sqrt{\pi t\,|q'(x_j(r,z))|\,(x_2(r,z)-x_1(r,z))}}$$

It is seen from formula (2.15) that, for long times, the perturbations decay as $1/\sqrt{t}$. Formula (2.15) becomes inexact when the roots $x_2$ and $x_1$ are close, that is,

$$x_2 \approx x_1 \approx bca/L(r) \tag{2.16}$$

$$z + 2k \approx aL(r)/b \quad \text{or} \quad 2 - z + 2k \approx aL(r)/b$$

In this case, the integrals

$$I_k = \int_{x_1}^{x_2} \frac{G_k(x,r,z)e^{iq(x)t}dx}{\sqrt{(x-x_1)(x_2-x)}}, \quad k = 1,2$$

make the main contribution.

Using the standard technique [3], we obtain

$$I_k \approx G_k(x_1,r,z)e^{iq(x_1)t}\int_{x_1}^{x_2}\frac{e^{iq'(x_1)(x-x_1)t}dx}{\sqrt{(x-x_1)(x_2-x)}} =$$

$$= G_k(x_1,r,z)e^{it\left(q(x_1)+\frac{1}{2}q'(x_1)(x_2-x_1)\right)}J_0(q'(x_1)(x_2-x_1)t) \tag{2.17}$$

The surfaces on which $x_2 = x_1$ are defined by equalities (2.16). It can be seen that these surfaces are hyperboloids of revolution. It follows from equality (2.17) that, close to these surfaces, the perturbations slowly decay. This fact can also be an obstacle for the propagation of radiowaves in the middle layer.

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